Programme 14

Multiple integration 1

Frames 1 to 83

Learning outcomes

When you have completed this Programme you will be able to:

- Evaluate double and triple integrals and apply them to the determination of the areas of plane figures and the volumes of solids
- Understand the role of the differential of a function of two or more real variables
- Determine exact differentials in two real variables and their integrals
- Evaluate the area enclosed by a closed curve by contour integration
- Evaluate line integrals and appreciate their properties
- Evaluate line integrals around closed curves within a simply connected region
- Link line integrals to integrals along the x-axis
- Link line integrals to integrals along a contour given in parametric form
- Discuss the dependence of a line integral between two points on the path of integration
- Determine exact differentials in three real variables and their integrals
- Demonstrate the validity and use of Green's theorem

Prerequisite: Engineering Mathematics (Fifth Edition)
Programme 23 Multiple integrals

Introduction

1

The introductory work on double and triple integrals was covered in detail in Programme 23 of *Engineering Mathematics (Fifth Edition)* and another look at the main points before launching forth on the current development could well be worth while.

You will no doubt recognise the following.

1 *Double integrals*

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy$$

is a double integral and is evaluated from the inside outwards, i.e.

$$\boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y)\,dx}^{\,①}\,dy}^{\,②}$$

A double integral is sometimes expressed in the form

$$\int_{y_1}^{y_2} dy \int_{x_1}^{x_2} f(x,y)\,dx$$

in which case, we evaluate from the right-hand end, i.e.

$$\int_{y_1}^{y_2} dy\ \boxed{\int_{x_1}^{x_2} f(x,y)\,dx}^{\,①}$$

then $$\boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\,dy}^{\,②}$$

2 *Triple integrals*

Triple integrals follow the same procedure.

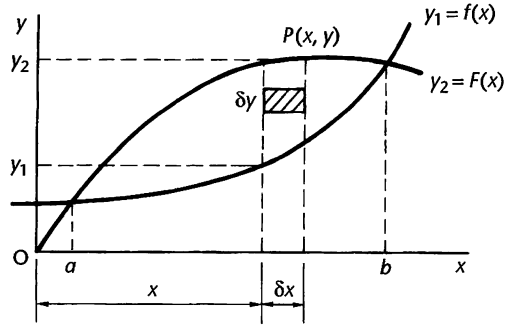
$\displaystyle\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz$ is evaluated in the order

$$\boxed{\int_{z_1}^{z_2} \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y,z)\,dx}^{\,①}\,dy}^{\,②}\,dz}^{\,③}$$

▶

3 *Applications*

(a) *Areas of plane figures*



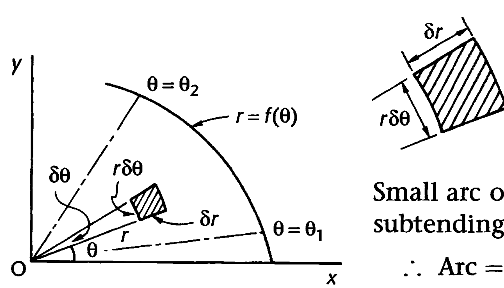
Area of element $\delta A = \delta x \delta y$

Area of strip $\approx \sum_{y=y_1}^{y=y_2} \delta x \delta y$

Area of all such strips $\approx \sum_{x=a}^{x=b} \left\{ \sum_{y=y_1}^{y=y_2} \delta x \delta y \right\}$

If $\delta x \to 0$ and $\delta y \to 0$, $A = \int_a^b \int_{y_1}^{y_2} \mathrm{d}y\,\mathrm{d}x$

(b) *Areas of plane figures bounded by a polar curve* $r = f(\theta)$ and radius vectors at $\theta = \theta_1$ and $\theta = \theta_2$



Small arc of circle of radius r, subtending angle $\delta\theta$ at centre.

$\therefore$ Arc $= r\delta\theta$

Area of element $\delta A \approx r\delta\theta\,\delta r$

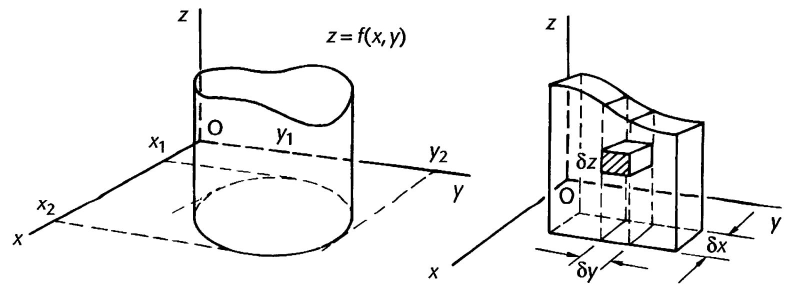
Area of thin sector $\approx \sum_{r=0}^{r=f(\theta)} r\,\delta\theta\,\delta r$

$\therefore$ Total area of all such sectors $\approx \sum_{\theta=\theta_1}^{\theta=\theta_2} \left\{ \sum_{r=0}^{r=f(\theta)} r\,\delta r\,\delta\theta \right\}$

$\therefore$ If $\delta r \to 0$ and $\delta\theta \to 0$, $A = \int_{\theta_1}^{\theta_2} \int_0^{r=f(\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta$

▶

(c) *Volume of solids*



Volume of element $\delta V = \delta x\, \delta y\, \delta z$

Volume of column $\approx \displaystyle\sum_{z=0}^{z=f(x,\, y)} \delta x\, \delta y\, \delta z$

Volume of slice $\approx \displaystyle\sum_{y=y_1}^{y=y_2} \left\{ \sum_{z=0}^{z=f(x,\, y)} \delta x\, \delta y\, \delta z \right\}$

$\therefore$ Total volume $V \approx$ sum of all such slices

i.e. $\quad V \approx \displaystyle\sum_{x=x_1}^{x=x_2} \sum_{y=y_1}^{y=y_2} \sum_{z=0}^{z=f(x,\, y)} \delta x\, \delta y\, \delta z$

Then, if $\delta x \to 0$, $\delta y \to 0$, $\delta z \to 0$,

$$V = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{0}^{z=f(x,\, y)} \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$$

If $z = f(x, y)$, this becomes

$$V = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\, \mathrm{d}y\, \mathrm{d}x$$

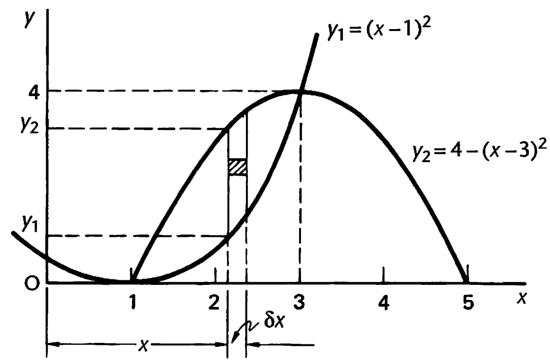
2

4 *Revision examples* As a means of 'warming up', let us work through one or two straightforward examples on the previous work.

Example 1

Find the area of the plane figure bounded by the curves $y_1 = (x-1)^2$ and $y_2 = 4-(x-3)^2$.

The first thing, as always, is to sketch the curves – each of which is a parabola – and to determine their points of intersection.



Points of intersection: $(x-1)^2 = 4-(x-3)^2$

$x^2 - 2x + 1 = 4 - x^2 + 6x - 9$ i.e. $x^2 - 4x + 3 = 0$

$\therefore\ (x-1)(x-3) = 0 \quad \therefore\ x = 1$ or $x = 3$.

Now we have all the information to determine the required area, which is

....

3

$$\boxed{A = 2\tfrac{2}{3} \text{ square units}}$$

Because

$$A = \int_{x=1}^{x=3}\int_{y_1}^{y_2} \mathrm{d}y\,\mathrm{d}x = \int_{x=1}^{x=3}\int_{y=(x-1)^2}^{y=4-(x-3)^2} \mathrm{d}y\,\mathrm{d}x$$

$$= \int_1^3 \{4-(x-3)^2-(x-1)^2\}\,\mathrm{d}x = -2\int_1^3 (x^2-4x+3)\,\mathrm{d}x$$

$$= -2\left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 = 2\tfrac{2}{3} \text{ square units}$$

Now for another.

▶

Example 2

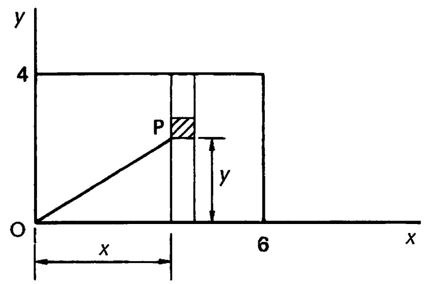
A rectangular plate is bounded by the x and y axes and the lines $x = 6$ and $y = 4$. The thickness t of the plate at any point is proportional to the square of the distance of the point from the origin. Determine the total volume of the plate.

First of all draw the figure and build up the appropriate double integral. Do not evaluate it yet. The expression is therefore

$$V = \dots\dots\dots\dots$$

4

$$\boxed{V = \int_{x=0}^{x=6} \int_{y=0}^{y=4} k(x^2 + y^2)\,\mathrm{d}y\,\mathrm{d}x}$$



Thickness t of plate at P is

$$t = k\,\mathrm{OP}^2 = k(x^2 + y^2)$$

Element of area $= \delta y\,\delta x$

$\therefore$ Element of volume at P $\approx k(x^2 + y^2)\,\delta y\,\delta x$

$$\therefore \text{ Total volume } V = \int_{x=0}^{x=6} \int_{y=0}^{y=4} k(x^2 + y^2)\,\mathrm{d}y\,\mathrm{d}x$$

Now we can evaluate the integral. We start from the inside with $\displaystyle\int_{y=2}^{y=4} k(x^2 + y^2)\,\mathrm{d}y$, remembering that for this integral (volume of the strip) x is constant. This gives $\dots\dots\dots\dots$

5

$$\boxed{k\left(4x^2 + \frac{64}{3}\right)}$$

Because

$$k\int_0^4 (x^2 + y^2)\,\mathrm{d}y = k\left[x^2 y + \frac{y^3}{3}\right]_{y=0}^{y=4} = k\left(4x^2 + \frac{64}{3}\right)$$

Then $\displaystyle V = k\int_0^6 \left(4x^2 + \frac{64}{3}\right)\mathrm{d}x = \dots\dots\dots\dots$

6

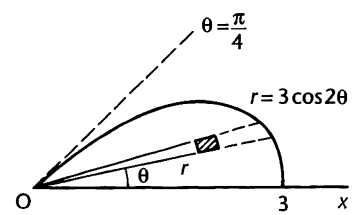
$$\boxed{V = 416k \text{ cubic units}}$$

That was easy enough. Notice that an alternative interpretation of this problem could be that of a uniform lamina with a variable density $\rho = k(x^2 + y^2)$ at any point (x, y). Now for one in polar coordinates.

Example 3

Express as a double integral the area enclosed by one loop of the curve $r = 3\cos 2\theta$ and evaluate the integral (refer to *Engineering Mathematics (Fifth Edition)*, Programme 22, Frame 11).

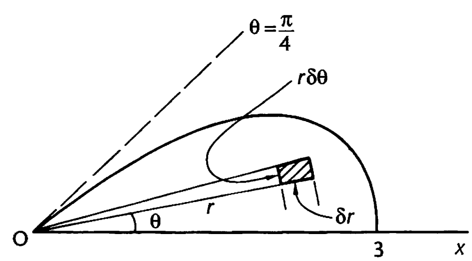
Consider the half loop shown.



First set up the double integral which is ....

7

$$\boxed{A = \int_{\theta=0}^{\theta=\pi/4}\int_{r=0}^{r=3\cos 2\theta} r\,dr\,d\theta}$$



Area of element $= r\,\delta r\,\delta\theta$

$\therefore$ Area of sector $\approx \displaystyle\sum_{r=0}^{r=3\cos 2\theta} r\,\delta r\,\delta\theta$

$\therefore$ Area of half loop $\approx \displaystyle\sum_{\theta=0}^{\theta=\pi/4}\sum_{r=0}^{r=3\cos 2\theta} r\,\delta r\,\delta\theta$

If $\delta r \to 0$ and $\delta\theta \to 0$,

$$A = \int_{\theta=0}^{\theta=\pi/4}\int_{r=0}^{r=3\cos 2\theta} r\,dr\,d\theta$$

Now finish it off to find the area of the whole loop, which is

....

$$\boxed{\frac{9\pi}{8} \text{ square units}}$$

Because

$$\begin{aligned}
A &= \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=3\cos 2\theta} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{3\cos 2\theta} \mathrm{d}\theta \\
&= \frac{9}{2}\int_0^{\pi/4} \cos^2 2\theta\,\mathrm{d}\theta \\
&= \frac{9}{4}\int_0^{\pi/4} (1+\cos 4\theta)\,\mathrm{d}\theta \\
&= \frac{9}{4}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/4} \\
&= \frac{9\pi}{16}
\end{aligned}$$

This is the area of a half loop.

$$\text{Required area} = \frac{9\pi}{8} \text{ square units}$$

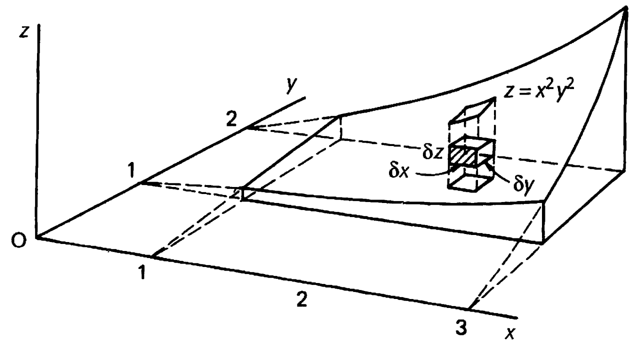
Now here is another.

Example 4

Find the volume of the solid bounded by the planes $z = 0$, $x = 1$, $x = 3$, $y = 1$, $y = 2$ and the surface $z = x^2y^2$.

As always, we start off by sketching the figure. When you have done that, check the result with the next frame.

9



We now build up the integral which will give us the volume of the solid.

Element of volume $\delta V = \delta x\,\delta y\,\delta z$

$$\text{Volume of column} \approx \sum_{z=0}^{z=x^2y^2} \delta x\,\delta y\,\delta z$$

$$\text{Volume of slice} \approx \sum_{y=1}^{y=2} \left\{ \sum_{z=0}^{z=x^2y^2} \delta x\,\delta y\,\delta z \right\}$$

$$\text{Volume of solid} \approx \sum_{x=1}^{x=3} \left\{ \sum_{y=1}^{y=2} \sum_{z=0}^{z=x^2y^2} \delta x\,\delta y\,\delta z \right\}$$

When $\delta x \to 0$, $\delta y \to 0$, $\delta z \to 0$,

$$V = \int_{x=1}^{x=3} \int_{y=1}^{y=2} \int_{z=0}^{z=x^2y^2} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Evaluating this, $V = \ldots\ldots\ldots\ldots$

10

$$\boxed{V = 20\tfrac{2}{9} \text{ cubic units}}$$

Because, starting with the innermost integral

$$V = \int_{x=1}^{x=3} \int_{y=1}^{y=2} \Big[z \Big]_0^{x^2y^2} \mathrm{d}y\,\mathrm{d}x = \int_1^3 \int_1^2 x^2y^2\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_1^3 \left[\frac{x^2\,y^3}{3} \right]_{y=1}^{y=2} \mathrm{d}x \qquad = \int_1^3 \frac{7x^2}{3}\,\mathrm{d}x = 20\tfrac{2}{9}$$

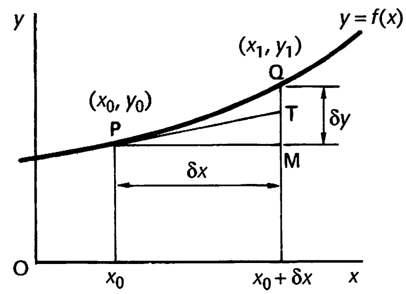
Now that we have revised the basics, let us move on to something rather different

Differentials

11

It is convenient in various branches of the calculus to denote small increases in value of a variable by the use of *differentials*. The method is particularly useful in dealing with the effects of small finite changes and shortens the writing of calculus expressions.

We are already familiar with the diagram from which finite changes δy and δx in a function $y = f(x)$ are depicted.



The increase in y from P to Q $= \text{MQ} = \delta y = f(x_0 + \delta x) - f(x_0)$

If PT is the tangent at P, then $\text{MQ} = \text{MT} + \text{TQ}$. Also $\dfrac{\text{MT}}{\delta x} = f'(x_0)$

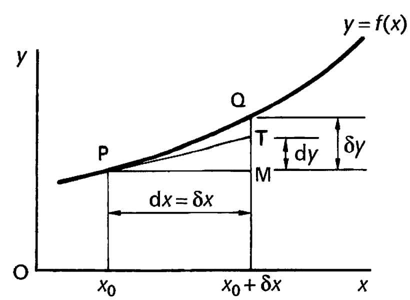
$\therefore\ \text{MT} = f'(x_0)\delta x$

$\therefore\ \text{MQ} = \delta y = f'(x_0) \cdot \delta x + \text{TQ}$

and, if Q is close to P, then $\delta y \approx f'(x_0)\delta x$

We define the differentials dy and dx as finite quantities such that

$$\mathrm{d}y = f'(x_0)\,\mathrm{d}x$$



Note that the differentials dy and dx are finite quantities – not necessarily zero – and can therefore exist alone.

Note too that $\mathrm{d}x = \delta x$.

From the diagram, we can see that

δy is the increase in y as we move from P to Q along the curve.
dy is the increase in y as we move from P to T along the tangent.

As Q approaches P, the difference between δy and dy decreases to zero. The use of differentials simplifies the writing of many relationships and is based on the general statement $dy = f'(x)\,dx$.

For example

(a) $y = x^5$ then $dy = 5x^4\,dx$
(b) $y = \sin 3x$ then $dy = 3\cos 3x\,dx$
(c) $y = e^{4x}$ then $dy = 4e^{4x}\,dx$
(d) $y = \cosh 2x$ then $dy = 2\sinh 2x\,dx$

Note that when the left-hand side is a differential dy the right-hand side must also contain a differential. Remember therefore to include the 'dx' on the right-hand side.

The product and quotient rules can also be expressed in differentials.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{becomes} \quad d(uv) = u\,dv + v\,du$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \quad \text{becomes} \quad d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

So, if $y = e^{2x}\sin 4x$, $dy = \ldots\ldots\ldots\ldots$

and if $y = \dfrac{\cos 2t}{t^2}$ $dy = \ldots\ldots\ldots\ldots$

12

$$\boxed{\begin{aligned} &y = e^{2x}\sin 4x, && dy = 2e^{2x}(2\cos 4x + \sin 4x)\,dx \\ &y = \frac{\cos 2t}{t^2}, && dy = -\frac{2}{t^3}\{t\sin 2t + \cos 2t\}\,dt \end{aligned}}$$

That was easy enough. Let us now consider a function of two independent variables, $z = f(x, y)$.

If $z = f(x, y)$ then $z + \delta z = f(x + \delta x, y + \delta y)$

$$\therefore \quad \delta z = f(x + \delta x, y + \delta y) - f(x, y)$$

Expanding δz in terms of δx and δy, gives

$\delta z = A\delta x + B\delta y +$ higher powers of δx and δy,
where A and B are functions of x and y.

If y remains constant, i.e. $\delta y = 0$, then

$$\delta z = A\,\delta x + \text{higher powers of } \delta x \quad \therefore \quad \frac{\delta z}{\delta x} \approx A$$

$$\therefore \text{ If } \delta x \to 0, \text{ then } A = \frac{\partial z}{\partial x}$$

▶

Similarly, if x remains constant, i.e. $\delta x = 0$, then

$$\delta z = B\,\delta y + \text{higher powers of } \delta y \quad \therefore \quad \frac{\delta z}{\delta y} \approx B$$

$$\therefore \text{ If } \delta y \to 0, \text{ then } B = \frac{\partial z}{\partial y}$$

$$\therefore \quad \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y + \text{higher powers of small quantities}$$

$$\therefore \quad \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

In terms of differentials, this result can be written

$$\text{If } z = f(x, y). \text{ then } \mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y$$

The result can be extended to functions of more than two independent variables.

$$\text{If } z = f(x, y, w), \quad \mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y + \frac{\partial z}{\partial w}\mathrm{d}w$$

Make a note of these results in differential form as shown.

Exercise

Determine the differential dz for each of the following functions.

1 $z = x^2 + y^2$

2 $z = x^3 \sin 2y$

3 $z = (2x - 1)\,e^{3y}$

4 $z = x^2 + 2y^2 + 3w^2$

5 $z = x^3y^2w.$

Finish all five and then check the results.

13

1	$\mathrm{d}z = 2(x\,\mathrm{d}x + y\,\mathrm{d}y)$
2	$\mathrm{d}z = x^2(3\sin 2y\,\mathrm{d}x + 2x\cos 2y\,\mathrm{d}y)$
3	$\mathrm{d}z = e^{3y}\{2\,\mathrm{d}x + (6x - 3)\mathrm{d}y\}$
4	$\mathrm{d}z = 2(x\,\mathrm{d}x + 2y\,\mathrm{d}y + 3w\,\mathrm{d}w)$
5	$\mathrm{d}z = x^2y(3yw\,\mathrm{d}x + 2xw\,\mathrm{d}y + xy\,\mathrm{d}w)$

Now move on

14

Exact differential

We have just established that if $z = f(x, y)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

We now work in reverse.

Any expression $dz = P\,dx + Q\,dy$, where P and Q are functions of x and y, is an *exact differential* if it can be integrated to determine z.

$$\therefore\ P = \frac{\partial z}{\partial x} \quad \text{and} \quad Q = \frac{\partial z}{\partial y}\ .$$

Now $\dfrac{\partial P}{\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$ and $\dfrac{\partial Q}{\partial x} = \dfrac{\partial^2 z}{\partial x\,\partial y}$ and we know that $\dfrac{\partial^2 z}{\partial y\,\partial x} = \dfrac{\partial^2 z}{\partial x\,\partial y}$.

Therefore, for dz to be an exact differential $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ and this is the test we apply.

Example 1

$dz = (3x^2 + 4y^2)\,dx + 8xy\,dy$.

If we compare the right-hand side with $P\,dx + Q\,dy$, then

$$P = 3x^2 + 4y^2 \quad \therefore\ \frac{\partial P}{\partial y} = 8y$$

$$Q = 8xy \quad \therefore\ \frac{\partial Q}{\partial x} = 8y$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore\ dz \text{ is an exact differential}$$

Similarly, we can test this one.

Example 2

$dz = (1 + 8xy)\,dx + 5x^2\,dy$.

From this we find ....

15

$\boxed{dz \text{ is } \textit{not} \text{ an exact differential}}$

Because $dz = (1 + 8xy)\,dx + 5x^2\,dy$

$$\therefore\ P = 1 + 8xy \quad \therefore\ \frac{\partial P}{\partial y} = 8x$$

$$Q = 5x^2 \quad \therefore\ \frac{\partial Q}{\partial x} = 10x$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \quad \therefore\ dz \text{ is not an exact differential.}$$

▶

Exercise

Determine whether each of the following is an exact differential.

1 $\mathrm{d}z = 4x^3y^3\,\mathrm{d}x + 3x^4y^2\,\mathrm{d}y$

2 $\mathrm{d}z = (4x^3y + 2xy^3)\,\mathrm{d}x + (x^4 + 3x^2y^2)\,\mathrm{d}y$

3 $\mathrm{d}z = (15y^2e^{3x} + 2xy^2)\,\mathrm{d}x + (10ye^{3x} + x^2y)\,\mathrm{d}y$

4 $\mathrm{d}z = (3x^2e^{2y} - 2y^2e^{3x})\,\mathrm{d}x + (2x^3e^{2y} - 2ye^{3x})\,\mathrm{d}y$

5 $\mathrm{d}z = (4y^3\cos 4x + 3x^2\cos 2y)\,\mathrm{d}x + (3y^2\sin 4x - 2x^3\sin 2y)\,\mathrm{d}y$.

16

1 Yes	**2** Yes	**3** No	**4** No	**5** Yes

We have just tested whether certain expressions are, in fact, exact differentials – and we said previously that, by definition, an exact differential can be integrated. But how exactly do we go about it? The following examples will show.

Integration of exact differentials

$$\mathrm{d}z = P\,\mathrm{d}x + Q\,\mathrm{d}y \quad \text{where} \quad P = \frac{\partial z}{\partial x} \quad \text{and} \quad Q = \frac{\partial z}{\partial y}$$

$$\therefore \; z = \int P\,\mathrm{d}x \quad \text{and also} \quad z = \int Q\,\mathrm{d}y$$

Example 1

$\mathrm{d}z = (2xy + 6x)\,\mathrm{d}x + (x^2 + 2y^3)\,\mathrm{d}y$.

$$P = \frac{\partial z}{\partial x} = 2xy + 6x \qquad \therefore \; z = \int (2xy + 6x)\,\mathrm{d}x$$

$\therefore \; z = x^2y + 3x^2 + f(y)$ where $f(y)$ is an arbitrary function of y only, and is akin to the constant of integration in a normal integral.

$$\text{Also} \;\; Q = \frac{\partial z}{\partial y} = x^2 + 2y^3 \qquad \therefore \; z = \int (x^2 + 2y^3)\,\mathrm{d}y$$

$$\therefore \; z = \ldots\ldots\ldots\ldots$$

17

$$\boxed{z = x^2y + \frac{y^4}{2} + F(x) \text{ where } F(x) \text{ is an arbitrary function of } x \text{ only}}$$

So the two results tell us

$$z = x^2y + 3x^2 + f(y) \tag{1}$$

and $$z = x^2y + \frac{y^4}{2} + F(x) \tag{2}$$

For these two expressions to represent the same function, then

$f(y)$ in (1) must be $\frac{y^4}{2}$ already in (2)

and $F(x)$ in (2) must be $3x^2$ already in (1)

$$\therefore\ z = x^2y + 3x^2 + \frac{y^4}{2}$$

Example 2

Integrate $dz = (8e^{4x} + 2xy^2)\,dx + (4\cos 4y + 2x^2y)\,dy$.

Argue through the working in just the same way, from which we obtain

$z = \ldots\ldots\ldots\ldots$

18

$$\boxed{z = 2e^{4x} + x^2y^2 + \sin 4y}$$

Here it is. $dz = (8e^{4x} + 2xy^2)\,dx + (4\cos 4y + 2x^2y)\,dy$

$$P = \frac{\partial z}{\partial x} = 8e^{4x} + 2xy^2 \qquad \therefore\ z = \int (8e^{4x} + 2xy^2)dx$$

$$\therefore\ z = 2e^{4x} + x^2y^2 + f(y) \tag{1}$$

$$Q = \frac{\partial z}{\partial y} = 4\cos 4y + 2x^2y \qquad \therefore\ z = \int (4\cos 4y + 2x^2y)\,dy$$

$$\therefore\ z = \sin 4y + x^2y^2 + F(x) \tag{2}$$

For (1) and (2) to agree, $f(y) = \sin 4y$ and $F(x) = 2e^{4x}$

$$\therefore\ z = 2\,e^{4x} + x^2y^2 + \sin 4y$$

They are all done in the same way, so you will have no difficulty with the short exercise that follows. *On you go.*

Exercise

Integrate the following exact differentials to obtain the function z.

1. $dz = (6x^2 + 8xy^3)\,dx + (12x^2y^2 + 12y^3)\,dy$
2. $dz = (3x^2 + 2xy + y^2)\,dx + (x^2 + 2xy + 3y^2)\,dy$
3. $dz = 2(y+1)e^{2x}\,dx + (e^{2x} - 2y)\,dy$
4. $dz = (3y^2\cos 3x - 3\sin 3x)\,dx + (2y\sin 3x + 4)\,dy$
5. $dz = (\sinh y + y\sinh x)dx + (x\cosh y + \cosh x)\,dy$.

Finish all five before checking with the next frame.

19

1	$z = 2x^3 + 4x^2y^3 + 3y^4$
2	$z = x^3 + x^2y + xy^2 + y^3$
3	$z = e^{2x}(1 + y) - y^2$
4	$z = y^2 \sin 3x + \cos 3x + 4y$
5	$z = x \sinh y + y \cosh x.$

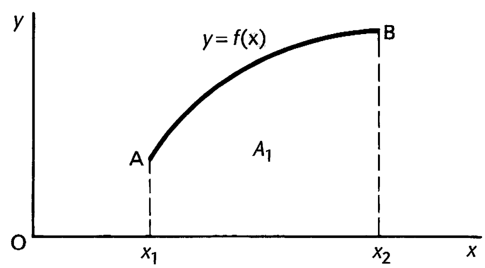
In the last one, of course, we find that the two expressions for z agree without any further addition of $f(y)$ or $F(x)$.

We shall be meeting exact differentials again later on, but for the moment let us deal with something different. On then to the next frame

Area enclosed by a closed curve

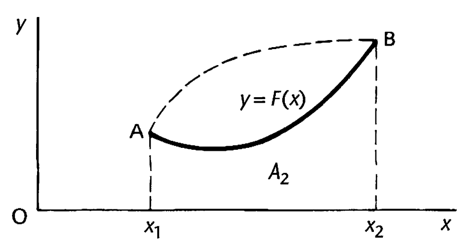
20

One of the earliest applications of integration is finding the area of a plane figure bounded by the x-axis, the curve $y = f(x)$ and ordinates at $x = x_1$ and $x = x_2$.



$$A_1 = \int_{x_1}^{x_2} y\,dx = \int_{x_1}^{x_2} f(x)\,dx$$

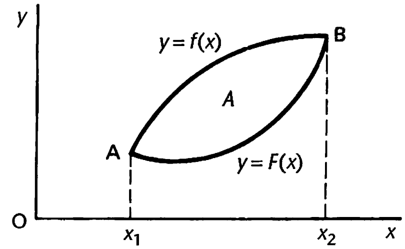
If points A and B are joined by another curve, $y = F(x)$



$$A_2 = \int_{x_1}^{x_2} F(x)\,dx$$

▶

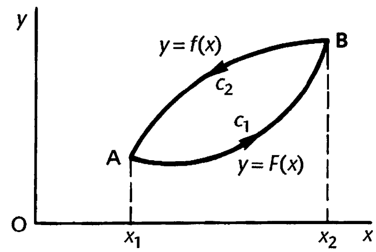
Combining the two figures, we have



$$A = A_1 - A_2$$

$$\therefore\ A = \int_{x_1}^{x_2} f(x)\,dx - \int_{x_1}^{x_2} F(x)\,dx$$

It is convenient on occasions to arrange the limits so that the integration follows the path round the enclosed area in a regular order.



For example

$\int_{x_1}^{x_2} F(x)\,dx$ gives A_2 as before, but integrating from B to A along c_2

with $y = f(x)$, i.e. $\int_{x_2}^{x_1} f(x)\,dx$, is the integral for A_1 with the sign

changed, i.e. $\int_{x_2}^{x_1} f(x)\,dx = -\int_{x_1}^{x_2} f(x)\,dx$

$\therefore$ The result $A = A_1 - A_2 = \int_{x_1}^{x_2} f(x)\,dx - \int_{x_1}^{x_2} F(x)\,dx$ becomes

$A = \ldots\ldots\ldots\ldots$

21

$$\boxed{A = -\int_{x_1}^{x_2} F(x)\,dx - \int_{x_2}^{x_1} f(x)\,dx}$$

i.e. $$A = -\left\{\int_{x_1}^{x_2} F(x)\,dx + \int_{x_2}^{x_1} f(x)\,dx\right\}$$

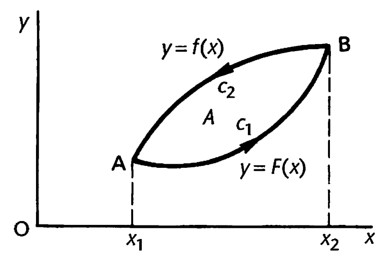
If we proceed round the boundary in an *anticlockwise manner*, the enclosed area is kept on the *left-hand side* and the resulting area is considered *positive*. If we proceed round the boundary in a *clockwise manner*, the enclosed area remains on the *right-hand side* and the resulting area is *negative*.

The final result above can be written in the form

$$A = -\oint y\,\mathrm{d}x$$

where the symbol $\oint$ indicates that the integral is to be evaluated round the closed boundary in the positive (i.e. anticlockwise) direction

$$\therefore\ A = -\oint y\,\mathrm{d}x = -\left\{ \underset{(\text{along } c_1)}{\int_{x_1}^{x_2} F(x)\,\mathrm{d}x} + \underset{(\text{along } c_2)}{\int_{x_2}^{x_1} f(x)\,\mathrm{d}x} \right\}$$



Let us apply this result to a very simple case.

Example 1

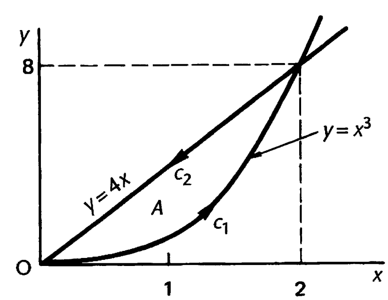
Determine the area enclosed by the graphs of $y = x^3$ and $y = 4x$ for $x \geq 0$.

First we need to know the points of intersection. These are

. . . .

22

$$\boxed{x = 0 \text{ and } x = 2}$$



We integrate in an anticlockwise manner

c_1: $y = x^3$, limits $x = 0$ to $x = 2$

c_2: $y = 4x$, limits $x = 2$ to $x = 0$.

$$A = -\oint y\,\mathrm{d}x = \ldots\ldots\ldots\ldots$$

23

$$\boxed{A = 4 \text{ square units}}$$

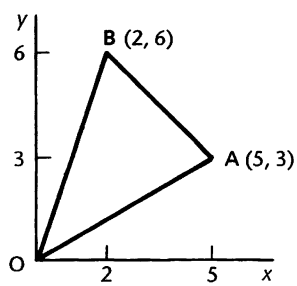
Because

$$A = -\oint y\,dx = -\left\{\int_0^2 x^3\,dx + \int_2^0 4x\,dx\right\}$$
$$= -\left\{\left[\frac{x^4}{4}\right]_0^2 + \left[2x^2\right]_2^0\right\} = 4$$

Another example.

Example 2

Find the area of the triangle with vertices (0, 0), (5, 3) and (2, 6).



The equation of

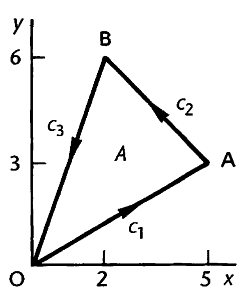
OA is ....

BA is ....

OB is ....

24

$$\boxed{\begin{array}{l} \text{OA is } y = \frac{3}{5}x \\ \text{BA is } y = 8 - x \\ \text{OB is } y = 3x \end{array}}$$



Then $A = -\oint y\,dx$

$= \ldots\ldots\ldots\ldots$

Write down the component integrals with appropriate limits.

25

$$\boxed{A = -\oint y\,dx = -\left\{\int_0^5 \frac{3}{5}x\,dx + \int_5^2 (8 - x)\,dx + \int_2^0 3x\,dx\right\}}$$

The limits chosen must progress the integration round the boundary of the figure in an *anticlockwise manner*. Finishing off the integration, we have

$A = \ldots\ldots\ldots\ldots$

26

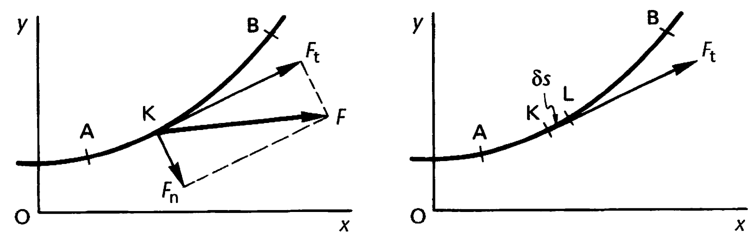
$$\boxed{A = 12 \text{ square units}}$$

The actual integration is easy enough.

The work we have just done leads us on to consider ***line integrals****, so let us make a fresh start in the next frame*

Line integrals

27



If a field exists in the x–y plane, producing a force F on a particle at K, then F can be resolved into two components

F_t along the tangent to the curve AB at K
F_n along the normal to the curve AB at K.

The work done in moving the particle through a small distance δs from K to L along the curve is then approximately $F_t\,\delta s$. So the total work done in moving a particle along the curve from A to B is given by

. . . .

28

$$\boxed{\underset{\delta s \to 0}{Lim} \sum F_t\,\delta s = \int F_t\,ds \text{ from A to B}}$$

This is normally written $\int_{AB} F_t\,ds$ where A and B are the end points of the curve, or as $\int_c F_t\,ds$ where the curve c connecting A and B is defined.

Such an integral thus formed is called a *line integral* since integration is carried out along the path of the particular curve c joining A and B.

$$\therefore\ I = \int_{AB} F_t\,ds = \int_c F_t\,ds$$

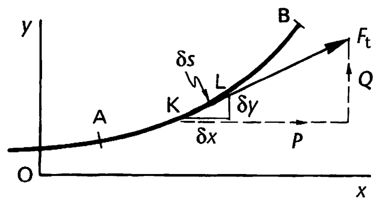
where c is the curve $y = f(x)$ between A (x_1, y_1) and B (x_2, y_2).

There is in fact an alternative form of the integral which is often useful, so let us also consider that

29

Alternative form of a line integral

It is often more convenient to integrate with respect to x or y than to take arc length as the variable.



If F_t has a component

P in the x-direction

Q in the y-direction

then the work done from K to L can be stated as $P\,\delta x + Q\,\delta y$.

$$\therefore \quad \int_{AB} F_t \, ds = \int_{AB} (P\,dx + Q\,dy)$$

where P and Q are functions of x and y.

In general then, the line integral can be expressed as

$$I = \int_c F_t \, ds = \int_c (P\,dx + Q\,dy)$$

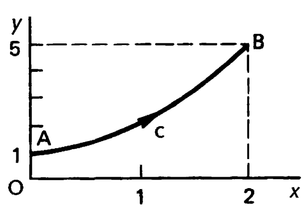
where c is the prescribed curve and F, or P and Q, are functions of x and y.

Make a note of these results – then we will apply them to one or two examples

30

Example 1

Evaluate $\int_c (x + 3y)dx$ from A (0, 1) to B (2, 5) along the curve $y = 1 + x^2$.



The line integral is of the form

$$\int_c (P\,dx + Q\,dy)$$

where, in this case, $Q = 0$ and c is the curve $y = 1 + x^2$.

It can be converted at once into an ordinary integral by substituting for y and applying the appropriate limits of x.

$$I = \int_c (P\,dx + Q\,dy) = \int_c (x + 3y)\,dx = \int_0^2 (x + 3 + 3x^2)\,dx$$

$$= \left[\frac{x^2}{2} + 3x + x^3 \right]_0^2 = 16$$

Now for another, so move on

31

Example 2

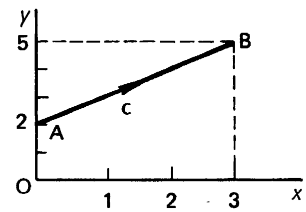
Evaluate $I = \int_c (x^2 + y)\,dx + (x - y^2)\,dy$ from A (0, 2) to B (3, 5) along the curve $y = 2 + x$.

$$I = \int_c (P\,dx + Q\,dy)$$

$$P = x^2 + y = x^2 + 2 + x = x^2 + x + 2$$

$$Q = x - y^2 = x - (4 + 4x + x^2)$$

$$= -(x^2 + 3x + 4)$$



Also $y = 2 + x \quad \therefore \; dy = dx$ and the limits are $x = 0$ to $x = 3$.

$$\therefore \; I = \ldots\ldots\ldots\ldots$$

32

$$\boxed{I = -15}$$

Because

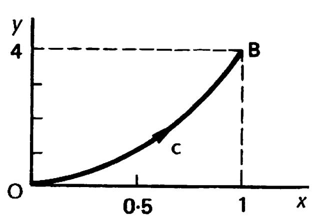
$$I = \int_0^3 \{(x^2 + x + 2)\,dx - (x^2 + 3x + 4)\,dx\}$$

$$\int_0^3 -(2x + 2)dx = \left[x^2 - 2x\right]_0^3 = -15$$

Here is another.

Example 3

Evaluate $I = \int_c \{(x^2 + 2y)\,dx + xy\,dy\}$ from O (0, 0) to B (1, 4) along the curve $y = 4x^2$.



In this case, c is the curve $y = 4x^2$.

$\therefore \; dy = 8x\,dx$

Substitute for y in the integral and apply the limits.

Then $I = \ldots\ldots\ldots\ldots$

Finish it off: it is quite straightforward.

33

$$\boxed{I = 9{\cdot}4}$$

Because

$$I = \int_c \{(x^2 + 2y)\,dx + xy\,dy\} \qquad y = 4x^2 \qquad \therefore\ dy = 8x\,dx$$

Also $x^2 + 2y = x^2 + 8x^2 = 9x^2$; $\quad xy = 4x^3$

$$\therefore\ I = \int_0^1 \{9x^2\,dx + 32x^4\,dx\} = \int_0^1 (9x^2 + 32x^4)\,dx = 9.4$$

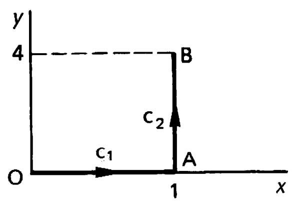
They are all done in very much the same way.

Move on for Example 4

34

Example 4

Evaluate $I = \int_c \{(x^2 + 2y)\,dx + xy\,dy\}$ from O (0, 0) to A (1, 0) along line $y = 0$ and then from A (1, 0) to B (1, 4) along the line $x = 1$.



(1) OA: c_1 is the line $y = 0$ $\therefore$ $dy = 0$. Substituting $y = 0$ and $dy = 0$ in the given integral gives

$$I_{OA} = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

(2) AB: Here c_2 is the line $x = 1$ $\quad\therefore$ $dx = 0$

$$\therefore\ I_{AB} = \ldots\ldots\ldots\ldots$$

35

$$\boxed{I_{AB} = 8}$$

Because

$$I_{AB} = \int_0^4 \{(1 + 2y)(0) + y\,dy\}$$

$$= \int_0^4 y\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^4 = 8$$

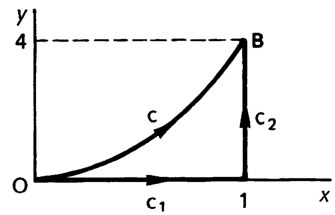
Then $I = I_{OA} + I_{AB} = \frac{1}{3} + 8 = 8\frac{1}{3}$ $\quad\therefore\ I = 8\frac{1}{3}$

If we now look back to Examples 3 and 4 just completed, we find that we have evaluated the same integral between the same two end points, but …………

36

along different paths of integration

If we combine the two diagrams, we have



where c is the curve $y = 4x^2$ and $c_1 + c_2$ are the lines $y = 0$ and $x = 1$. The results obtained were

$I_c = 9\frac{2}{5}$ and $I_{c_1+c_2} = 8\frac{1}{3}$

Notice therefore that integration along two distinct paths joining the same two end points does not necessarily give the same results.

37

Let us pause here a moment and list the main properties of line integrals.

Properties of line integrals

1 $\displaystyle\int_c F\,ds = \int_c \{P\,dx + Q\,dy\}$

2 $\displaystyle\int_{AB} F\,ds = -\int_{BA} F\,ds$ and $\displaystyle\int_{AB} \{P\,dx + Q\,dy\} = -\int_{BA} \{P\,dx + Q\,dy\}$

i.e. the sign of a line integral is reversed when the direction of the integration along the path is reversed.

3 (a) For a path of integration parallel to the y-axis, i.e. $x = k$,

$dx = 0. \quad \therefore \displaystyle\int_c P\,dx = 0 \quad \therefore I_c = \int_c Q\,dy.$

(b) For a path of integration parallel to the x-axis, i.e. $y = k$,

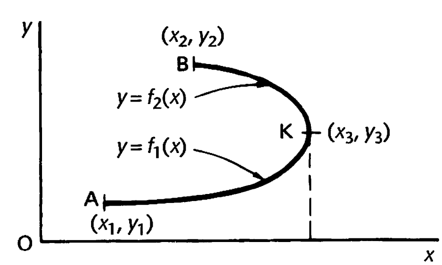
$dy = 0. \quad \therefore \displaystyle\int_c Q\,dy = 0 \quad \therefore I_c = \int_c P\,dx.$

4 If the path of integration c joining A to B is divided into two parts AK and KB, then $I_c = I_{AB} = I_{AK} + I_{KB}$.

5 In all cases, the function $y = f(x)$ that describes the path of integration involved must be continuous and single-valued – or dealt with as in item **6** below.

6 If the function $y = f(x)$ that describes the path of integration c is not single-valued for part of its extent, the path is divided into two sections.

$y = f_1(x)$ from A to K
$y = f_2(x)$ from K to B.

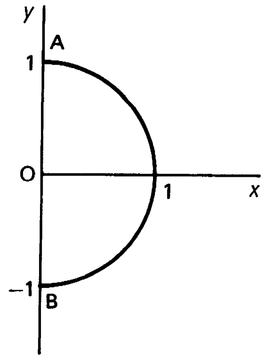


Make a note of this list for future reference and revision

38

Example

Evaluate $I = \int_c (x+y)\,dx$ from A (0, 1) to B (0, − 1) along the semicircle $x^2 + y^2 = 1$ for $x \geq 0$.



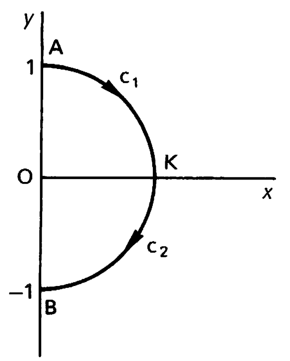
The first thing we notice is that

....

39

$$\boxed{\text{the function } y = f(x) \text{ that describes the path of integration c is } \textit{not} \text{ single-valued}}$$

For any value of x, $y = \pm\sqrt{1-x^2}$. Therefore, we divide c into two parts



(1) $y = \sqrt{1-x^2}$ from A to K

(2) $y = -\sqrt{1-x^2}$ from K to B.

As usual, $I = \int_c (P\,dx + Q\,dy)$ and in this particular case, Q = ...

40

$$\boxed{Q = 0}$$

$$\therefore\ I = \int_c P\,dx = \int_0^1 \left(x + \sqrt{1-x^2}\right) dx + \int_1^0 (x - \sqrt{1-x^2})\,dx$$

$$= \int_0^1 (x + \sqrt{1-x^2} - x + \sqrt{1-x^2})\,dx = 2\int_0^1 \sqrt{1-x^2}\,dx$$

Now substitute $x = \sin\theta$ and finish it off.

$I =$ ....

41

$$\boxed{I = \frac{\pi}{2}}$$

Because

$$I = 2\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x \quad x = \sin\theta \quad \therefore\ \mathrm{d}x = \cos\theta\,\mathrm{d}\theta$$

$$\sqrt{1-x^2} = \cos\theta$$

Limits: $x = 0,\ \theta = 0;\ \ x = 1,\ \theta = \dfrac{\pi}{2}$

$$\therefore\ I = 2\int_0^{\pi/2} \cos^2\theta\,\mathrm{d}\theta = \int_0^{\pi/2} (1+\cos 2\theta)\,\mathrm{d}\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

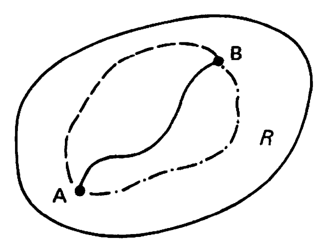
$$= \frac{\pi}{2}$$

Now let us extend this line of development a stage further.

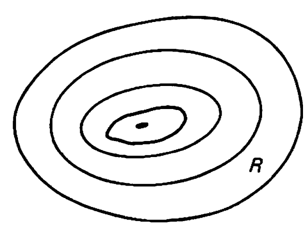
Regions enclosed by closed curves

42

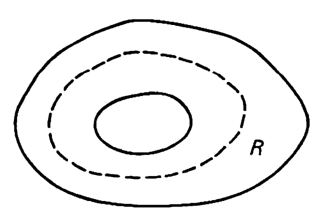
A region is said to be *simply connected* if a path joining A and B can be deformed to coincide with any other line joining A and B without going outside the region.



Another definition is that a region is simply connected if any closed path in the region can be contracted to a single point without leaving the region.



Clearly, this would not be satisfied in the case where the region R contains one or more 'holes'.

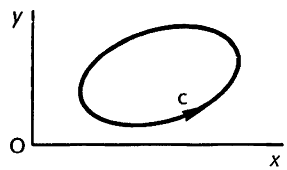


The closed curves involved in problems in this Programme all relate to simply connected regions, so no difficulties will arise.

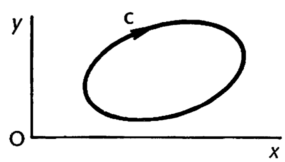
43

Line integrals round a closed curve

We have already introduced the symbol $\oint$ to indicate that an integral is to be evaluated round a closed curve in the positive (anticlockwise) direction.

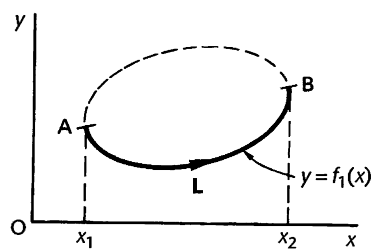


Positive direction (anticlockwise) line integral denoted by $\oint$.

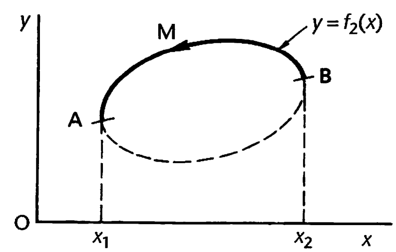


Negative direction (clockwise) line integral denoted by $-\oint$.

With a closed curve, the y-values on the path c cannot be single-valued. Therefore, we divide the path into two or more parts and treat each separately.



(1) Use $y = f_1(x)$ for ALB



(2) Use $y = f_2(x)$ for BMA.

Unless specially required otherwise, we always proceed round the closed curve in an ....

44

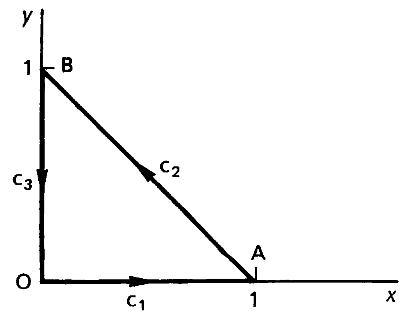
anticlockwise direction

Example 1

Evaluate the line integral $I = \oint_c (x^2\,dx - 2xy\,dy)$ where c comprises the three sides of the triangle joining O (0, 0), A (1, 0) and B (0, 1).

First draw the diagram and mark in c_1, c_2 and c_3, the proposed directions of integration. Do just that.

45



The three sections of the path of integration must be arranged in an anticlockwise manner round the figure. Now we deal with each part separately.

(a) OA: c_1 is the line $y = 0 \quad \therefore \; dy = 0$.

Then $I = \oint(x^2\,dx - 2xy\,dy)$ for this part becomes

$$I_1 = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \qquad \therefore \; I_1 = \frac{1}{3}$$

(b) AB: c_2 is the line $y = 1 - x \quad \therefore \; dy = -dx$

$I_2 = \ldots\ldots\ldots\ldots$ (evaluate it)

46

$$\boxed{I_2 = -\tfrac{2}{3}}$$

Because c_2 is the line $y = 1 - x \quad \therefore \; dy = -dx$.

$$I_2 = \int_1^0 \{x^2\,dx + 2x(1-x)\,dx\} = \int_1^0 (x^2 + 2x - 2x^2)\,dx$$

$$= \int_1^0 (2x - x^2)dx = \left[x^2 - \frac{x^3}{3}\right]_1^0 = -\frac{2}{3} \quad \therefore \; I_2 = -\frac{2}{3}$$

Note that anticlockwise progression is obtained by arranging the limits in the appropriate order.

Now we have to determine I_3 for BO.

(c) BO: c_3 is the line $x = 0$

$I_3 = \ldots\ldots\ldots\ldots$

47

$$\boxed{I_3 = 0}$$

Because for c_3, $\quad x = 0 \quad \therefore \; dx = 0 \quad \therefore \; I_3 = \int 0\,dy = 0 \quad \therefore \; I_3 = 0$

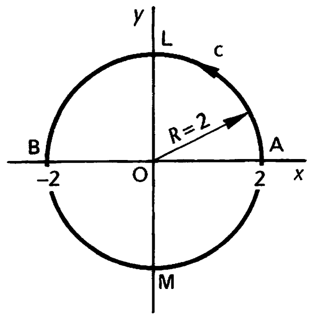
Finally, $I = I_1 + I_2 + I_3 = \frac{1}{3} - \frac{2}{3} + 0 = -\frac{1}{3} \qquad \therefore \; I = -\frac{1}{3}$

Let us work through another example.

▶

Example 2

Evaluate $\oint_c y\,dx$ when c is the circle $x^2 + y^2 = 4$.



$$x^2 + y^2 = 4 \quad \therefore \ y = \pm\sqrt{4 - x^2}$$

y is thus not single-valued. Therefore use $y = \sqrt{4 - x^2}$ for ALB between $x = 2$ and $x = -2$ and $y = -\sqrt{4 - x^2}$ for BMA between $x = -2$ and $x = 2$.

$$\therefore \ I = \int_2^{-2} \sqrt{4 - x^2}\,dx + \int_{-2}^{2} \{-\sqrt{4 - x^2}\}\,dx$$

$$= 2\int_2^{-2} \sqrt{4 - x^2}\,dx = -2\int_{-2}^{2} \sqrt{4 - x^2}\,dx$$

$$= -4\int_0^{2} \sqrt{4 - x^2}\,dx.$$

To evaluate this integral, substitute $x = 2\sin\theta$ and finish it off.

$I = \ldots\ldots\ldots\ldots$

48

$$\boxed{I = -4\pi}$$

Because

$$x = 2\sin\theta \quad \therefore \ dx = 2\cos\theta\,d\theta \quad \therefore \ \sqrt{4 - x^2} = 2\cos\theta$$

limits: $x = 0, \ \theta = 0; \ x = 2, \ \theta = \dfrac{\pi}{2}$

$$\therefore \ I = -4\int_0^{\pi/2} 2\cos\theta\, 2\cos\theta\,d\theta = -16\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= -8\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta = -8\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = -4\pi$$

Now for one more

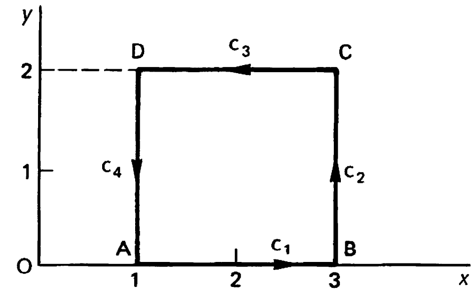
Example 3

Evaluate $I = \oint_c \{xy\,dx + (1 + y^2)\,dy\}$ where c is the boundary of the rectangle joining A (1, 0), B (3, 0), C (3, 2) and D (1, 2).

First draw the diagram and insert c_1, c_2, c_3, c_4.

That gives $\ldots\ldots\ldots\ldots$

49



Now evaluate I_1 for AB; I_2 for BC; I_3 for CD; I_4 for DA; and finally I.

Complete the working and then check with the next frame

50

$$\boxed{I_1 = 0; \quad I_2 = 4\tfrac{2}{3}; \quad I_3 = -8; \quad I_4 = -4\tfrac{2}{3}; \quad I = -8}$$

Here is the complete working.

$$I = \oint_c \{xy\,dx + (1 + y^2)\,dy\}$$

(a) AB: c_1 is $y = 0 \quad \therefore\ dy = 0 \quad \therefore\ I_1 = 0$

(b) BC: c_2 is $x = 3 \quad \therefore\ dx = 0$

$$\therefore\ I_2 = \int_0^2 (1 + y^2)dy = \left[y + \frac{y^3}{3}\right]_0^2 = 4\tfrac{2}{3} \qquad \therefore\ I_2 = 4\tfrac{2}{3}$$

(c) CD: c_3 is $y = 2 \quad \therefore\ dy = 0$

$$\therefore\ I_3 = \int_3^1 2x\,dx = \left[x^2\right]_3^1 = -8 \qquad \therefore\ I_3 = -8$$

(d) DA: c_4 is $x = 1 \quad \therefore\ dx = 0$

$$\therefore\ I_4 = \int_2^0 (1 + y^2)\,dy = \left[y + \frac{y^3}{3}\right]_2^0 = -4\tfrac{2}{3} \quad \therefore\ I_4 = -4\tfrac{2}{3}$$

Finally

$$I = I_1 + I_2 + I_3 + I_4$$
$$= 0 + 4\tfrac{2}{3} - 8 - 4\tfrac{2}{3} = -8 \qquad \therefore\ I = -8$$

Remember that, unless we are directed otherwise, we always proceed round the closed boundary in an anticlockwise manner.

On now to the next piece of work

51 Line integral with respect to arc length

We have already established that

$$I = \int_{AB} F_t \, ds = \int_{AB} \{P \, dx + Q \, dy\}$$

where F_t denoted the tangential force along the curve c at the sample point K (x, y).

The same kind of integral can, of course, relate to any function $f(x, y)$ which is a function of the position of a point on the stated curve, so that $I = \int_c f(x, y) \, ds$.

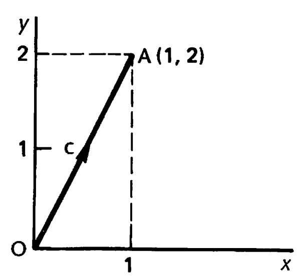
This can readily be converted into an integral in terms of x. (Refer to *Engineering Mathematics (Fifth Edition)*, Programme 19, Frame 30.)

$$I = \int_c f(x, y) \, ds = \int_c f(x, y) \frac{ds}{dx} dx \quad \text{where} \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\therefore \int_c f(x, y) \, ds = \int_{x_1}^{x_2} f(x, y) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \qquad (1)$$

Example

Evaluate $I = \int_c (4x + 3xy) \, ds$ where c is the straight line joining O (0, 0) to A (1, 2).



c is the line $y = 2x \quad \therefore \dfrac{dy}{dx} = 2$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{5}$$

$$\therefore I = \int_{x=0}^{x=1} (4x + 3xy) \, ds = \int_0^1 (4x + 3xy)(\sqrt{5}) \, dx. \quad \text{But } y = 2x$$

$$\therefore I = \ldots\ldots\ldots\ldots$$

52

$$\boxed{I = 4\sqrt{5}}$$

Because

$$I = \int_0^1 (4x + 6x^2)(\sqrt{5})\,dx = 2\sqrt{5}\int_0^1 (2x + 3x^2)\,dx = 4\sqrt{5}$$

Try another.

The path length of the parabola defined by $y = x^2$ betwen the values $x = 0$ and $x = 2$ is given by the integral

$$I = \int_c ds = \ldots\ldots\ldots\ldots \quad \text{to 3 dp}$$

53

$$\boxed{3{\cdot}393 \text{ to 3 dp}}$$

Because

$$\begin{aligned} I &= \int_c ds = \int_{x=0}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)}\,dx \\ &= \int_{x=0}^{2} \sqrt{1 + 2x}\,dx \end{aligned}$$

Let $u = 1 + 2x$ so that $du = 2dx$ and so

$$\begin{aligned} I &= \int_{u=1}^{5} u^{1/2}\frac{du}{2} \\ &= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^5 \\ &= \frac{1}{3}\left(125^{1/2} - 1\right) \\ &= 3{\cdot}393 \text{ to 3 dp} \end{aligned}$$

Parametric equations

54

When x and y are expressed in parametric form, e.g. $x = f(t)$, $y = g(t)$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \therefore\ ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

and result (1) above becomes

$$I = \int_c f(x, y)\,ds = \int_{t_1}^{t_2} f(x, y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \qquad (2)$$

Make a note of results (1) and (2) for future use

55

Example

Evaluate $I = \oint_c 4xy\,ds$ where c is defined as the curve $x = \sin t$, $y = \cos t$ between $t = 0$ and $t = \dfrac{\pi}{4}$.

$$\text{We have } x = \sin t \quad \therefore \quad \frac{dx}{dt} = \cos t$$

$$y = \cos t \quad \therefore \quad \frac{dy}{dt} = -\sin t$$

$$\therefore \quad \frac{ds}{dt} = \ldots\ldots\ldots\ldots$$

56

$$\boxed{\frac{ds}{dt} = 1}$$

Because

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\therefore \quad I = \int_{t_1}^{t_2} f(x, y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{\pi/4} 4\sin t\,\cos t\,dt$$

$$= 2\int_0^{\pi/4} \sin 2t\,dt = -2\left[\frac{\cos 2t}{2}\right]_0^{\pi/4} = 1 \quad \therefore \quad I = 1$$

Dependence of the line integral on the path of integration

We saw earlier in the Programme that integration along two separate paths joining the same two end points does not necessarily give identical results.

With this in mind, let us investigate the following problem.

Example

Evaluate $I = \oint_c \{3x^2y^2\,dx + 2x^3y\,dy\}$ between O (0, 0) and A (2, 4)

(a) along c_1 i.e. $y = x^2$

(b) along c_2 i.e. $y = 2x$

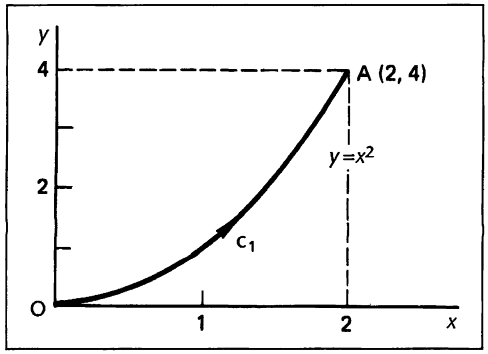
(c) along c_3 i.e. $x = 0$ from (0, 0) to (0, 4) and $y = 4$ from (0, 4) to (2, 4).

Let us concentrate on section (a).

First we draw the figure and insert relevant information.

This gives ....

57



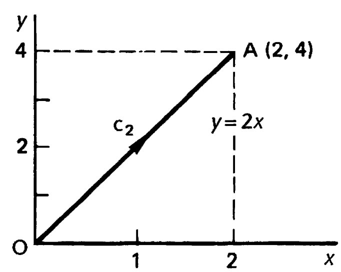
(a) $I = \int_c \{3x^2y^2\,dx + 2x^3y\,dy\}$

The path c_1 is $y = x^2$ $\quad \therefore \; dy = 2x\,dx$

$$\therefore \; I_1 = \int_0^2 \{3x^2x^4\,dx + 2x^3x^2 2x\,dx\} = \int_0^2 (3x^6 + 4x^4)\,dx$$

$$= \left[x^7\right]_0^2 = 128 \qquad \therefore \; I_1 = 128$$

(b) In (b), the path of integration changes to c_2, i.e. $y = 2x$



So, in this case,
$I_2 = \ldots\ldots\ldots\ldots$

58

$$\boxed{I_2 = 128}$$

Because with c_2, $\; y = 2x \quad \therefore \; dy = 2\,dx$

$$\therefore \; I_2 = \int_0^2 \{3x^2\,4x^2\,dx + 2x^3\,2x2\,dx\} = \int_0^2 20x^4\,dx$$

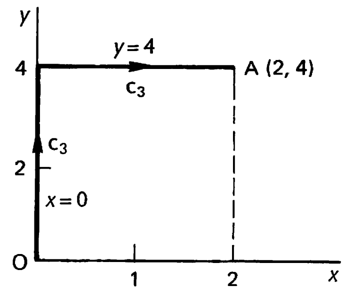
$$= 4\left[x^5\right]_0^2 = 128 \qquad \therefore \; I_2 = 128$$

(c) In the third case, the path c_3 is split
$x = 0$ from (0, 0) to (0, 4)
$y = 4$ from (0, 4) to (2, 4)
Sketch the diagram and determine I_3.

$I_3 = \ldots\ldots\ldots\ldots$

59

$$\boxed{I_3 = 128}$$



From (0, 0) to (0, 4) $x = 0 \quad \therefore \ \mathrm{d}x = 0 \quad \therefore \ I_{3a} = 0$

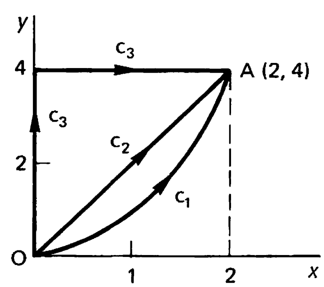
From (0, 4) to (2, 4) $y = 4 \quad \therefore \ \mathrm{d}y = 0 \quad \therefore \ I_{3b} = 48\int_0^2 x^2 \mathrm{d}x = 128$

$$\therefore \ I_3 = 128$$

On to the next frame

60

In the example we have just worked through, we took three different paths and in each case, the line integral produced the same result. It appears, therefore, that in this case, the value of the integral is independent of the path of integration taken.



How then does this integral perhaps differ from those of previous cases?

Let us investigate

61

We have been dealing with $I = \int_c \{3x^2y^2 \, \mathrm{d}x + 2x^3y \, \mathrm{d}y\}$

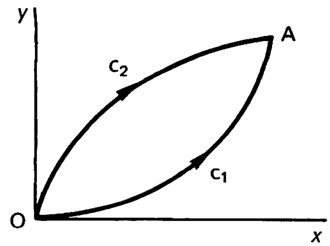
On reflection, we see that the integrand $3x^2y^2 \, \mathrm{d}x + 2x^3y \, \mathrm{d}y$ is of the form $P \, \mathrm{d}x + Q \, \mathrm{d}y$ which we have met before and that it is, in fact, an *exact differential* of the function $z = x^3y^2$, because

$$\frac{\partial z}{\partial x} = 3x^2y^2 \quad \text{and} \quad \frac{\partial z}{\partial y} = 2x^3y$$

Provided P, Q and their first partial derivatives are finite and continuous at all points inside and on any closed curve, this always happens. If the integrand of the given integral is seen to be an *exact differential*, then the value of the line integral is *independent of the path taken and depends only on the coordinates of the two end points*

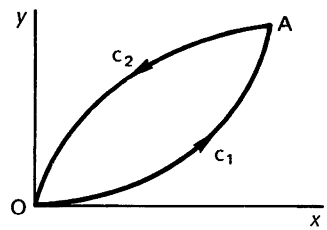
Make a note of this. It is important

62



If $I = \int_c \{P\,dx + Q\,dy\}$ and $(P\,dx + Q\,dy)$ is an exact differential, then

$I_{c_1} = I_{c_2}$



If we reverse the direction of c_2, then

$I_{c_1} = -I_{c_2}$

i.e. $I_{c_1} + I_{c_2} = 0$

Hence, *if $(P\,dx + Q\,dy)$ is an exact differential, then the integration taken round a closed curve is zero.*

$\therefore$ If $(P\,dx + Q\,dy)$ is an exact differential, $\oint(P\,dx + Q\,dy) = 0$

63

Example 1

Evaluate $I = \int_c \{3y\,dx + (3x + 2y)\,dy\}$ from A (1, 2) to B (3, 5).

No path is given, so the integrand is probably an exact differential of some function $z = f(x, y)$. In fact $\dfrac{\partial P}{\partial y} = 3 = \dfrac{\partial Q}{\partial x}$.

We have already dealt with the integration of exact differentials, so there is no difficulty. Compare with $I = \int_c \{P\,dx + Q\,dy\}$.

$$P = \frac{\partial z}{\partial x} = 3y \qquad \therefore\ z = \int 3y\,dx = 3xy + f(y) \qquad (1)$$

$$Q = \frac{\partial z}{\partial y} = 3x + 2y \quad \therefore\ z = \int (3x + 2y)\,dy = 3xy + y^2 + F(x) \qquad (2)$$

For (1) and (2) to agree

$f(y) = \ldots\ldots\ldots\ldots$ and $F(x) = \ldots\ldots\ldots\ldots$

64

$$\boxed{f(y) = y^2; \quad F(x) = 0}$$

Hence $z = 3xy + y^2$

$$\therefore \; I = \int_c \{3y\,dx + (3x + 2y)\,dy\} = \int_{(1,\,2)}^{(3,\,5)} d(3xy + y^2)$$

$$= \Big[3xy + y^2\Big]_{(1,\,2)}^{(3,\,5)}$$

$$= (45 + 25) - (6 + 4)$$

$$= 60$$

Example 2

Evaluate $I = \int_c \{(x^2 + ye^x)\,dx + (e^x + y)\,dy\}$ between A (0, 1) and B (1, 2).

As before, compare with $\int_c \{P\,dx + Q\,dy\}$.

$$P = \frac{\partial z}{\partial x} = x^2 + ye^x \quad \therefore \; z = \ldots\ldots\ldots\ldots$$

$$Q = \frac{\partial z}{\partial y} = e^x + y \quad \therefore \; z = \ldots\ldots\ldots\ldots$$

Continue the working and complete the evaluation.

When you have finished, check the result with the next frame

65

$$\boxed{\begin{aligned} z &= \frac{x^3}{3} + ye^x + f(y) \\ z &= ye^x + \frac{y^2}{2} + F(x) \end{aligned}}$$

For these expressions to agree, $\quad f(y) = \dfrac{y^2}{2}; \quad F(x) = \dfrac{x^3}{3}$

$$\text{Then } I = \left[\frac{x^3}{3} + ye^x + \frac{y^2}{2}\right]_{(0,\,1)}^{(1,\,2)}$$

$$= \frac{5}{6} + 2e$$

So the main points are that, if $(P\,dx + Q\,dy)$ is an exact differential

(a) $I = \int_c (P\,dx + Q\,dy)$ is independent of the path of integration

(b) $I = \int_c (P\,dx + Q\,dy)$ is zero when c is a closed curve.

On to the next frame

Exact differentials in three independent variables 66

A line integral in space naturally involves three independent variables, but the method is very much like that for two independent variables.

$\mathrm{d}w = P\mathrm{d}x + Q\mathrm{d}y + R\,\mathrm{d}z$ is an exact differential of $w = f(x, y, z)$

$$\text{if} \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}; \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}; \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$$

If the test is successful, then

(a) $\int_c (P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z)$ is independent of the path of integration

(b) $\oint_c (P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z)$ is zero when c is a closed curve.

Example

Verify that $\mathrm{d}w = (3x^2yz + 6x)\mathrm{d}x + (x^3z - 8y)\mathrm{d}y + (x^3y + 1)\mathrm{d}z$ is an exact differential and hence evaluate $\int_c \mathrm{d}w$ from A (1, 2, 4) to B (2, 1, 3).

First check that $\mathrm{d}w$ is an exact differential by finding the partial derivatives above, when $P = 3x^2yz + 6x$; $Q = x^3z - 8y$; and $R = x^3y + 1$.

We have ....

67

$$\boxed{\begin{array}{lll} \dfrac{\partial P}{\partial y} = 3x^2z; & \dfrac{\partial Q}{\partial x} = 3x^2z & \therefore \ \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \\ \dfrac{\partial P}{\partial z} = 3x^2y; & \dfrac{\partial R}{\partial x} = 3x^2y & \therefore \ \dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x} \\ \dfrac{\partial R}{\partial y} = x^3; & \dfrac{\partial Q}{\partial z} = x^3 & \therefore \ \dfrac{\partial R}{\partial y} = \dfrac{\partial Q}{\partial z} \\ \multicolumn{3}{c}{\therefore \ \mathrm{d}w \text{ is an exact differential}} \end{array}}$$

Now to find w. $\quad P = \dfrac{\partial z}{\partial x}; \quad Q = \dfrac{\partial z}{\partial y}; \quad R = \dfrac{\partial w}{\partial z}$

$$\therefore \ \frac{\partial w}{\partial x} = 3x^2yz + 6x \qquad \therefore \ w = \int (3x^2yz + 6x)\mathrm{d}x$$
$$= x^3yz + 3x^2 + f(y, z)$$
$$\frac{\partial w}{\partial y} = x^3z - 8y \qquad \therefore \ w = \int (x^3z - 8y)\,\mathrm{d}y$$
$$= x^3zy - 4y^2 + F(x, z)$$
$$\frac{\partial w}{\partial z} = x^3y + 1 \qquad \therefore \ w = \int (x^3y + 1)\,\mathrm{d}z$$
$$= x^3yz + z + g(x, y)$$

For these three expressions for z to agree

$f(y, z) = \ldots\ldots\ldots\ldots$; $\quad F(x, z) = \ldots\ldots\ldots\ldots$; $\quad g(x, y) = \ldots\ldots\ldots\ldots$

68

$$\boxed{f(y,z) = -4y^2; \quad F(x,z) = z; \quad g(x,y) = 3x^2}$$

$$\therefore \; w = x^3yz + 3x^2 - 4y^2 + z$$

$$\therefore \; I = \Big[x^3yz + 3x^2 - 4y^2 + z\Big]_{(1,2,4)}^{(2,1,3)}$$

$$= \ldots\ldots\ldots\ldots$$

69

$$\boxed{I = 36}$$

Because

$$I = \Big[x^3yz + 3x^2 - 4y^2 + z\Big]_{(1,2,4)}^{(2,1,3)}$$

$$= (24 + 12 - 4 + 3) - (8 + 3 - 16 + 4) = 36$$

The extension to line integrals in space is thus quite straightforward.

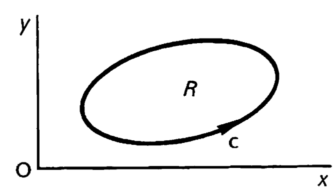
Finally, we have a theorem that can be very helpful on occasions and which links up with the work we have been doing.

It is important, so let us start a new section

Green's theorem

70

Let P and Q be two functions of x and y that are, along with their first partial derivatives, finite and continuous inside and on the boundary c of a region R in the x–y plane.



If the first partial derivatives are continuous within the region and on the boundary, then Green's theorem states that

$$\int_R\int\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \mathrm{d}x\,\mathrm{d}y = -\oint_c (P\,\mathrm{d}x + Q\,\mathrm{d}y)$$

That is, a double integral over the plane region R can be transformed into a line integral over the boundary c of the region – and the action is reversible.

Let us see how it works.

Example 1

Evaluate $I = \oint_c \{(2x - y)\,dx + (2y + x)\,dy\}$ around the boundary c of the ellipse $x^2 + 9y^2 = 16$.

The integral is of the form $I = \oint_c \{P\,dx + Q\,dy\}$ where

$$P = 2x - y \quad \therefore \quad \frac{\partial P}{\partial y} = -1$$

$$\text{and } Q = 2y + x \quad \therefore \quad \frac{\partial Q}{\partial x} = 1.$$

$$\therefore \; I = -\int_R\int\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$$

$$= -\int_R\int(-1 - 1)\,dx\,dy$$

$$= 2\int_R\int dx\,dy$$

But $\int_R\int dx\,dy$ over any closed region gives ....

71

the area of the figure

In this case, then, $I = 2A$ where A is the area of the ellipse

$$x^2 + 9y^2 = 16 \quad \text{i.e.} \quad \frac{x^2}{16} + \frac{9y^2}{16} = 1$$

$$\therefore \; a = 4; \; b = \frac{4}{3}$$

$$\therefore \; A = \pi ab = \frac{16\pi}{3}$$

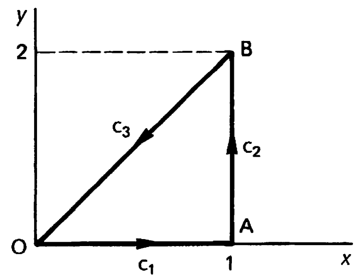
$$\therefore \; I = 2A = \frac{32\pi}{3}$$

To demonstrate the advantage of Green's theorem, let us work through the next example (a) by the method of line integrals, and (b) by applying Green's theorem.

▶

Example 2

Evaluate $I = \oint_c \{(2x + y)\,dx + (3x - 2y)\,dy\}$ taken in anticlockwise manner round the triangle with vertices at O (0, 0), A (1, 0) and B (1, 2).



$$I = \oint_c \{(2x + y)\,dx + (3x - 2y)\,dy\}$$

(a) *By the method of line integrals*

There are clearly three stages with c_1, c_2, c_3. Work through the complete evaluation to determine the value of I. It will be good revision.

When you have finished, check the result with the solution in the next frame

72

$$\boxed{I = 2}$$

(a) (1) c_1 is $y = 0 \quad \therefore \; dy = 0$

$$\therefore \; I_1 = \int_0^1 2x\,dx = \Big[x^2\Big]_0^1 = 1 \quad \therefore \; I_1 = 1$$

(2) c_2 is $x = 1 \quad \therefore \; dx = 0$

$$\therefore \; I_2 = \int_0^2 (3 - 2y)\,dy = \Big[3y - y^2\Big]_0^2 = 2 \quad \therefore \; I_2 = 2$$

(3) c_3 is $y = 2x \quad \therefore \; dy = 2\,dx$

$$\therefore \; I_3 = \int_1^0 \{4x\,dx + (3x - 4x)2\,dx\}$$

$$= \int_1^0 2x\,dx = \Big[x^2\Big]_1^0 = -1 \qquad \therefore \; I_3 = -1$$

$$I = I_1 + I_2 + I_3 = 1 + 2 + (-1) = 2 \quad \therefore \; I = 2$$

Now we will do the same problem by applying Green's theorem, so move on

73

(b) *By Green's theorem*

$$I = \oint_c \{(2x + y)\,dx + (3x - 2y)\,dy\}$$

$$P = 2x + y \quad \therefore \quad \frac{\partial P}{\partial y} = 1; \quad Q = 3x - 2y \quad \therefore \quad \frac{\partial Q}{\partial x} = 3$$

$$I = -\int_R\int\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx\,dy$$

Finish it off. $I = \dots\dots\dots\dots$

74

$$\boxed{I = 2}$$

Because

$$I = -\int_R\int(1-3)\,dx\,dy$$
$$= 2\int_R\int dx\,dy = 2A$$
$$= 2 \times \text{ the area of the triangle}$$
$$= 2 \times 1 = 2 \quad \therefore \quad I = 2$$

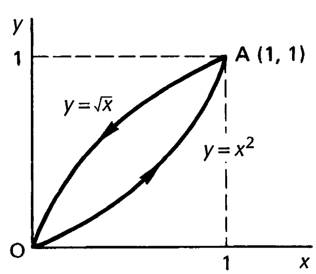
Application of Green's theorem is not always the quickest method. It is useful, however, to have both methods available. If you have not already done so, make a note of Green's theorem.

$$\int_R\int\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx\,dy = -\oint_c (P\,dx + Q\,dy)$$

75

Example 3

Evaluate the line integral $I = \oint_c \{xy\,dx + (2x - y)\,dy\}$ round the region bounded by the curves $y = x^2$ and $x = y^2$ by the use of Green's theorem.



Points of intersection are O (0, 0) and A (1, 1). P and Q are known, so there is no difficulty.

Complete the working.

$I = \dots\dots\dots\dots$

76

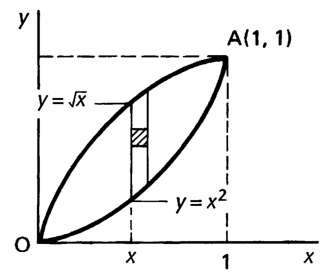
$$\boxed{I = \frac{31}{60}}$$

Here is the working.

$$I = \oint_c \{xy\,dx + (2x - y)\,dy\}$$

$$\oint_c \{P\,dx + Q\,dy\} = -\int_R\!\int \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$$

$$P = xy \quad \therefore \quad \frac{\partial P}{\partial y} = x; \quad Q = 2x - y \quad \therefore \quad \frac{\partial Q}{\partial x} = 2$$



$$I = -\int_R\!\int (x-2)\,dx\,dy$$

$$= -\int_0^1 \int_{y=x^2}^{y=\sqrt{x}} (x-2)\,dy\,dx$$

$$= -\int_0^1 (x-2)\Big[y\Big]_{x^2}^{\sqrt{x}}\,dx$$

$$\therefore \; I = -\int_0^1 (x-2)(\sqrt{x} - x^2)\,dx$$

$$= -\int_0^1 (x^{3/2} - x^3 - 2x^{1/2} + 2x^2)\,dx$$

$$= -\left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4 - \frac{4}{3}x^{3/2} + \frac{2}{3}x^3\right]_0^1 = \frac{31}{60}$$

Before we finally leave this section of the work, there is one more result to note.

In the special case when $P = y$ and $Q = -x$

$$\frac{\partial P}{\partial y} = 1 \quad \text{and} \quad \frac{\partial Q}{\partial x} = -1$$

Green's theorem then states

$$\int_R\!\int \{1 - (-1)\}\,dx\,dy = -\oint_c (P\,dx + Q\,dy)$$

i.e. $$2\int_R\!\int dx\,dy = -\oint_c (y\,dx - x\,dy)$$

$$= \oint_c (x\,dy - y\,dx)$$

Therefore, the area of the closed region

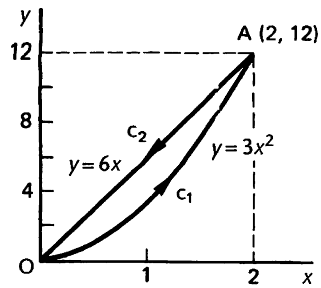
$$A = \int_R\!\int dx\,dy = \frac{1}{2}\oint_c (x\,dy - y\,dx)$$

Note this result in your record book. Then let us see an example

77

Example 1

Determine the area of the figure enclosed by $y = 3x^2$ and $y = 6x$.



Points of intersection:

$$3x^2 = 6x \quad \therefore \; x = 0 \text{ or } 2$$

$$\text{Area } A = \frac{1}{2}\oint_c (x\,dy - y\,dx)$$

We evaluate the integral in two parts, i.e. OA along c_1 and AO along c_2

$$2A = \int_{c_1 \text{ (along OA)}} (x\,dy - y\,dx) + \int_{c_2 \text{ (along AO)}} (x\,dy - y\,dx) = I_1 + I_2$$

I_1: c_1 is $y = 3x^2 \quad \therefore \; dy = 6x\,dx$

$$\therefore \; I_1 = \int_0^2 (6x^2\,dx - 3x^2\,dx) = \int_0^2 3x^2\,dx = \Big[x^3\Big]_0^2 = 8$$

$$\therefore \; I_1 = 8$$

Similarly, $I_2 = \ldots\ldots\ldots\ldots$

78

$$\boxed{I_2 = 0}$$

Because

c_2 is $y = 6x \quad \therefore \; dy = 6\,dx$

$$\therefore \; I_2 = \int_2^0 (6x\,dx - 6x\,dx) = 0 \quad \therefore \; I_2 = 0$$

$$\therefore \; I = I_1 + I_2 = 8 + 0 = 8 \quad \therefore \; A = 4 \text{ square units}$$

Finally, here is one for you to do entirely on your own.

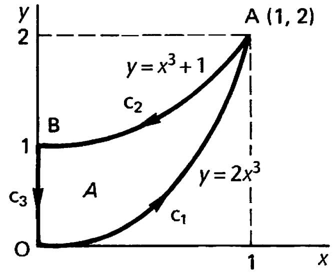
Example 2

Determine the area bounded by the curves $y = 2x^3$, $y = x^3 + 1$ and the axis $x = 0$ for $x \geq 0$.

Complete the working and see if you agree with the working in the next frame

79

Here it is.



$y = 2x^3; \quad y = x^3 + 1; \quad x = 0$

Point of intersection

$2x^3 = x^3 + 1 \quad \therefore \; x^3 = 1 \quad \therefore \; x = 1$

Area $A = \dfrac{1}{2}\oint_c (x\,dy - y\,dx)$

$\therefore \; 2A = \oint_c (x\,dy - y\,dx)$

(a) OA: c_1 is $y = 2x^3 \quad \therefore \; dy = 6x^2\,dx$

$$\therefore \; I_1 = \int_{c_1} (x\,dy - y\,dx) = \int_0^1 (6x^3\,dx - 2x^3\,dx)$$

$$= \int_0^1 4x^3\,dx = \Big[x^4\Big]_0^1 = 1 \qquad \therefore \; I_1 = 1$$

(b) AB: c_2 is $y = x^3 + 1 \quad \therefore \; dy = 3x^2\,dx$

$$\therefore \; I_2 = \int_1^0 \{3x^3\,dx - (x^3 + 1)\,dx\} = \int_1^0 (2x^3 - 1)\,dx$$

$$= \left[\frac{x^4}{2} - x\right]_1^0 = -(\tfrac{1}{2} - 1) = \tfrac{1}{2} \qquad \therefore \; I_2 = \tfrac{1}{2}$$

(c) BO: c_3 is $x = 0 \quad \therefore \; dx = 0$

$$I_3 = \int_{y=1}^{y=0} (x\,dy - y\,dx) = 0 \qquad \therefore \; I_3 = 0$$

$\therefore \; 2A = I = I_1 + I_2 + I_3 = 1 + \frac{1}{2} + 0 = 1\frac{1}{2}$

$\therefore \; A = \frac{3}{4}$ square units

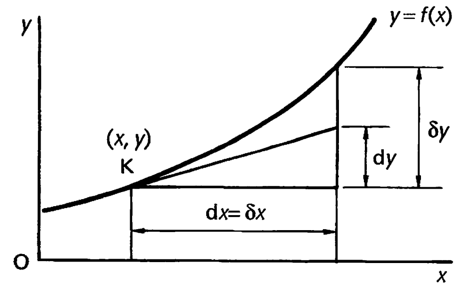
And that brings this Programme to an end. We have covered some important topics, so check down the **Revision summary** and the **Can You?** checklist that follow and revise any part of the text if necessary, before working through the **Test exercise**. The **Further problems** provide an opportunity for additional practice.

Revision summary 14

80

1 *Differentials* dy and dx

(a)



$$\mathrm{d}y = f'(x)\,\mathrm{d}x$$

(b) If $z = f(x, y)$, $\quad \mathrm{d}z = \dfrac{\partial z}{\partial x}\,\mathrm{d}x + \dfrac{\partial z}{\partial y}\,\mathrm{d}y$

If $z = f(x, y, w)$, $\quad \mathrm{d}z = \dfrac{\partial z}{\partial x}\,\mathrm{d}x + \dfrac{\partial z}{\partial y}\,\mathrm{d}y + \dfrac{\partial z}{\partial w}\,\mathrm{d}w.$

(c) $\mathrm{d}z = P\,\mathrm{d}x + Q\,\mathrm{d}y$, where P and Q are functions of x and y, is an *exact differential* if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

2 *Line integrals* – definition

$$I = \int_c f(x, y)\,\mathrm{d}s = \int_c (P\,\mathrm{d}x + Q\,\mathrm{d}y)$$

3 *Properties of line integrals*

(a) Sign of line integral is reversed when the direction of integration along the path is reversed.

(b) Path of integration parallel to y-axis, $\mathrm{d}x = 0 \;\therefore\; I_c = \displaystyle\int_c Q\,\mathrm{d}y$.

Path of integration parallel to x-axis, $\mathrm{d}y = 0 \;\therefore\; I_c = \displaystyle\int_c P\,\mathrm{d}x$.

(c) The y-values on the path of integration must be continuous and single-valued.

4 *Line of integral round a closed curve* $\oint$

Positive direction $\ointctrclockwise$ anticlockwise

Negative direction $\ointclockwise$ clockwise, i.e. $\ointclockwise = -\ointctrclockwise$.

▶

5 *Line integral related to arc length*

$$I = \int_{AB} F\,ds = \int_{AB} (P\,dx + Q\,dy)$$

$$= \int_{\substack{x_1 \\ c}}^{x_2} f(x, y)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

With parametric equations, x and y in terms of t,

$$I = \int_c f(x, y)\,ds = \int_{t_1}^{t_2} f(x, y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

6 *Dependence of line integral on path of integration*

In general, the value of the line integral depends on the particular path of integration.

7 *Exact differential*

If $P\,dx + Q\,dy$ is an exact differential where P, Q and their first derivatives are finite and continuous inside the simply connected region R

(a) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

(b) $I = \int_c (P\,dx + Q\,dy)$ is independent of the path of integration where c lies entirely within R

(c) $I = \oint_c (P\,dx + Q\,dy)$ is zero when c is a closed curve lying entirely within R.

8 *Exact differentials in three variables*

If $P\,dx + Q\,dy + R\,dz$ is an exact differential where P, Q, R and their first partial derivatives are finite and continuous inside a simply connected region containing path c

(a) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$; $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$; $\dfrac{\partial R}{\partial y} = \dfrac{\partial Q}{\partial z}$

(b) $\int_c (P\,dx + Q\,dy + R\,dz)$ is independent of the path of integration

(c) $\oint_c (P\,dx + Q\,dy + R\,dz)$ is zero when c is a closed curve.

9 *Green's theorem*

$$\oint_c (P\,dx + Q\,dy) = -\int_R\int \left\{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right\} dx\,dy$$

and, for a simple closed curve

$$\oint_c (x\,dy - y\,dx) = 2\int_R\int dx\,dy = 2A$$

where A is the area of the enclosed figure.

Can You?

Checklist 14

81

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Evaluate double and triple integrals and apply them to the determination of the areas of plane figures and the volumes of solids? 1 to 10

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Understand the role of the differential of a function of two or more real variables? 11 to 13

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine exact differentials in two real variables and their integrals? 14 to 19

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate the area enclosed by a closed curve by contour integration? 20 to 26

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate line integrals and appreciate their properties? 27 to 41

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate line integrals around closed curves within a simply connected region? 42 to 50

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Link line integrals to integrals along the x-axis? 51 to 53

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Link line integrals to integrals along a contour given in parametric form? 54 to 56

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Discuss the dependence of a line integral between two points on the path of integration? 56 to 65

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine exact differentials in three real variables and their integrals? 66 to 69

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Demonstrate the validity and use of Green's theorem? 70 to 79

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 14

82

1 Determine the differential dz of each of the following.
(a) $z = x^4 \cos 3y$; (b) $z = e^{2y} \sin 4x$; (c) $z = x^2yw^3$.

2 Determine which of the following are exact differentials and integrate where appropriate to determine z.
(a) $dz = (3x^2y^4 + 8x)\,dx + (4x^3y^3 - 15y^2)\,dy$
(b) $dz = (2x\cos 4y - 6\sin 3x)dx - 4(x^2 \sin 4y - 2y)\,dy$
(c) $dz = 3e^{3x}(1-y)\,dx + (e^{3x} + 3y^2)\,dy$.

3 Calculate the area of the triangle with vertices at O (0, 0), A (4, 2) and B (1, 5).

4 Evaluate the following.
(a) $I = \int_c \{(x^2 - 3y)\,dx + xy^2\,dy\}$ from A (1, 2) to B (2, 8) along the curve $y = 2x^2$.
(b) $I = \int_c (2x + y)\,dx$ from A (0, 1) to B (0, −1) along the semicircle $x^2 + y^2 = 1$ for $x \geq 0$.
(c) $I = \oint_c \{(1 + xy)\,dx + (1 + x^2)\,dy\}$ where c is the boundary of the rectangle joining A (1, 0), B (4, 0), C (4, 3) and D (1, 3).
(d) $I = \int_c 2xy\,ds$ where c is defined by the parametric equations $x = 4\cos\theta$, $y = 4\sin\theta$ between $\theta = 0$ and $\theta = \dfrac{\pi}{3}$.
(e) $I = \int_c \{(8xy + y^3)\,dx + (4x^2 + 3xy^2)\,dy\}$ from A(1, 3) to B(2, 1).
(f) $I = \oint_c \{(3x + y)\,dx + (y - 2x)\,dy\}$ round the boundary of the ellipse $x^2 + 4y^2 = 36$.

5 Apply Green's theorem to determine the area of the plane figure bounded by the curves $y = x^3$ and $y = \sqrt{x}$.

6 Verify that $dw = (2xyz + 2z - y^2)dx + (x^2z - 2yx)dy + (x^2y + 2x)dz$ is an exact differential and find the value of

$\int_c dw$ where

(a) c is the straight line joining (0, 0, 0) to (1, 1, 1)
(b) c is the curve of intersection of the unit sphere centred on the origin and the plane $x + y + z = 1$.

Further problems 14

83

1 Show that $I = \int_c \{xy^2w^2\,dx + x^2yw^2\,dy + x^2y^2w\,dw\}$ is independent of the path of integration c and evaluate the integral from A (1, 3, 2) to B (2, 4, 1).

2 Determine whether $dz = 3x^2(x^2 + y^2)\,dx + 2y(x^3 + y^4)\,dy$ is an exact differential. If so, determine z and hence evaluate $\int_c dz$ from A (1, 2) to B (2, 1).

3 Evaluate the line integral $I = \oint_c \left\{\dfrac{x\,dy - y\,dx}{x^2 + y^2 + 4}\right\}$ where c is the boundary of the segment formed by the arc of the circle $x^2 + y^2 = 4$ and the chord $y = 2 - x$ for $x \geq 0$.

4 Show that

$$I = \int_c \{(3x^2 \sin y + 2\sin 2x + y^3)\,dx + (x^3 \cos y + 3xy^2)\,dy\}$$

is independent of the path of integration and evaluate it from A (0, 0) to B $\left(\dfrac{\pi}{2}, \pi\right)$.

5 Evaluate the integral $I = \int_c xy\,ds$ where c is defined by the parametric equations $x = \cos^3 t$, $y = \sin^3 t$ from $t = 0$ to $t = \dfrac{\pi}{2}$.

6 Verify that $dz = \dfrac{x\,dx}{x^2 - y^2} - \dfrac{y\,dy}{x^2 - y^2}$ for $x^2 > y^2$ is an exact differential and evaluate $z = f(x, y)$ from A (3, 1) to B (5, 3).

7 The parametric equations of a circle, centre (1, 0) and radius 1, can be expressed as $x = 2\cos^2\theta$, $y = 2\cos\theta\sin\theta$.

Evaluate $I = \int_c \{(x + y)\,dx + x^2\,dy\}$ along the semicircle for which $y \geq 0$ from O (0, 0) to A (2, 0).

8 Evaluate $\oint_c \{x^3y^2\,dx + x^2y\,dy\}$ where c is the boundary of the region enclosed by the curve $y = 1 - x^2$, $x = 0$ and $y = 0$ in the first quadrant.

9 Use Green's theorem to evaluate

$$I = \oint_c \{(4x + y)\,dx + (3x - 2y)\,dy\}$$

where c is the boundary of the trapezium with vertices A (0, 1), B (5, 1), C (3, 3) and D (1, 3).

10 Evaluate $I = \int_c \{(3x^2y^2 + 2\cos 2x - 2xy)\,dx + (2x^3y + 8y - x^2)\,dy\}$

(a) along the curve $y = x^2 - x$ from A (0, 0) to B (2, 2)

(b) round the boundary of the quadrilateral joining the points (1, 0), (3, 1), (2, 3) and (0, 3)

▶

11 Verify that $\mathrm{d}w = \dfrac{y}{z}\mathrm{d}x + \dfrac{x}{z}\mathrm{d}y - \dfrac{xy}{z^2}\mathrm{d}z$ is an exact differential and find the value of

$$\int_c \mathrm{d}w$$

where c is the straight line joining (0, 0, 1) to (1, 2, 3) for either region $z > 0$ or $z < 0$.